MAT 162
Ch. 10 Review
Spring 2015

Name Key

1. Calculate the first four terms of the sequence and find its limit, if it exists.

$$a_n = \frac{2n}{n+1}$$

$\boxed{1,\ 4/3,\ 3/2,\ 8/5,\ \ldots}$ $\boxed{\lim_{n\to\infty} \frac{2n}{n+1} = 2}$

Find the limit of the sequence or state that it diverges. Justify your answers.

2. $a_n = 3\left(\frac{5}{4}\right)^n$

$\boxed{5/4 > 1}$
divergesc

3. $a_n = \left(\frac{7}{n}+81\right)^{\frac{1}{4}}$

$\lim_{n\to\infty} \left(\frac{7}{n}\overset{\to 0}{} + 81\right)^{1/4} = \boxed{3}$

4. $a_n = \frac{3n^3 - n}{1-2n^3}$

$\lim_{n\to\infty} \frac{3n^3 - n}{-2n^3+1} = \boxed{-3/2}$

5. $a_n = 1+(-1)^n$

$\boxed{0, 2, 0, 2, 0, 2, \ldots \text{ diverges}}$

Determine the convergence or divergence of the series. If the series converges, find the sum of the series. Justify your answers.

6. $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2}$

$(1 - \cancel{1/3}) + (1/2 - \cancel{1/4}) + (\cancel{1/3} - \cancel{1/5}) + (\cancel{1/4} - \cancel{1/6}) +$

$\left(\cancel{\frac{1}{n-2}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right)$

$\lim_{n\to\infty} \frac{3}{2} - \frac{1}{n+2}\overset{\to 0}{} = \boxed{3/2}$

7. $\sum_{n=0}^{\infty} 8\left(\frac{\pi}{e}\right)^n$

$\boxed{\pi/e > 1}$
diverges

8. $\displaystyle\sum_{n=0}^{\infty}\frac{6-2^n}{3^n}$

$\Sigma\, 6\left(\frac{1}{3}\right)^n - \Sigma\left(\frac{2}{3}\right)^n$

$\dfrac{6}{1-\frac{1}{3}} - \dfrac{1}{1-\frac{2}{3}}$

$\dfrac{6}{2/3} - \dfrac{1}{1/3} = 9 - 3 = \boxed{6}$

9. $\displaystyle\sum_{n=1}^{\infty}\frac{5n^4-3n+1}{8n-n^4}$

$\lim\limits_{n\to\infty}\left(\dfrac{5n^4-3n+1}{-n^4+8n}\right) = -5 \neq 0$

diverges n^{th} term test

Determine the convergence or divergence of the series using any method. Justify your answers.

10. $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n(n+2)}$

$\Sigma \frac{1}{n} \leftarrow$ div. p-series

$\lim\limits_{n\to\infty}\dfrac{\frac{n+1}{n^2+2n}}{\frac{1}{n}} = \lim\limits_{n\to\infty}\dfrac{n+1}{n^2+2n}\cdot\dfrac{n}{1}$

$= \lim\limits_{n\to\infty}\dfrac{n^2+n}{n^2+2n} = 1 > 0$ finite

so both diverge LCT

11. $\displaystyle\sum_{n=1}^{\infty}\frac{2^{2n}}{n!}$

Ratio

$\lim\limits_{n\to\infty}\dfrac{2^{2(n+1)}}{(n+1)!}\cdot\dfrac{n!}{2^{2n}}$

$\lim\limits_{n\to\infty}\dfrac{4}{n+1} = 0 < 1$

converges

12. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^2}-\frac{1}{\sqrt{n}}\right)$

conv > 1 div < 1

p-series

diverges

13. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(3n+1)^n}$

Root Test

$\displaystyle\sum_{n=1}^{\infty}\sqrt[n]{\left(\frac{1}{3n+1}\right)^n}$

$\lim\limits_{n\to\infty}\dfrac{1}{3n+1}$ $0 < 1$ converges

14. $\displaystyle\sum_{n=1}^{\infty}\frac{5}{n-2}$

DCT

$\Sigma \frac{5}{n}$ div p-series

$\dfrac{5}{n-2} > \dfrac{5}{n}$

so both diverge

15. $\displaystyle\sum_{n=1}^{\infty}\frac{(3n)!}{(n!)^3}$

Ratio

$\lim\limits_{n\to\infty}\dfrac{(3(n+1))!}{((n+1)!)^3}\cdot\dfrac{(n!)^3}{(3n)!}$

$\lim\limits_{n\to\infty}\dfrac{(3n+3)!\; n!\,n!\,n!}{(3n)!\;(n+1)!\,(n+1)!\,(n+1)!}$

$\lim\limits_{n\to\infty}\dfrac{(3n+1)(3n+2)(3n+3)}{(n+1)(n+1)(n+1)} = 27 > 1$ diverges

Determine whether the series diverges, converges absolutely, or converges conditionally.

16. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

$\sum \frac{1}{n\sqrt{n}} = \sum \frac{1}{n^{3/2}}$

$3/2 > 1$ $\boxed{\text{abs converges}}$

p-series

17. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n+3)}{n+10}$

$\lim_{n \to \infty} \frac{2n+3}{n+10} = 2 \neq 0$

$\boxed{\text{diverges - } n^{th} \text{ term test}}$

18. a) Does $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$ converge conditionally or absolutely?

AST

$\lim_{n \to \infty} \frac{1}{\sqrt{n+4}} = 0$ ✓

$\frac{1}{\sqrt{n+5}} < \frac{1}{\sqrt{n+4}}$ ✓ $\boxed{\text{conv cond.}}$

$\sum \frac{1}{\sqrt{n+4}}$ $\quad \sum \frac{1}{\sqrt{n}}$ div p-series

LCT $\frac{\frac{1}{\sqrt{n+4}}}{\frac{1}{\sqrt{n}}} = 1 > 0$ finite, $\boxed{\text{so both diverge}}$

b) Show that $|S - S_4| < \frac{1}{\sqrt{8}}$.

$\boxed{\frac{1}{2} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{7}}} + \frac{1}{\sqrt{8}}$

c) Find N such that S_n approximates S with an error less than $\frac{1}{4}$.

$\boxed{\frac{1}{2} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{8}} - \frac{1}{3} + \frac{1}{\sqrt{10}} - \frac{1}{\sqrt{11}} + \frac{1}{\sqrt{12}} - \frac{1}{\sqrt{13}} + \frac{1}{\sqrt{14}} - \frac{1}{\sqrt{15}}} \quad \frac{1}{4}$

$\boxed{N = 12}$

Find a power series representation by writing in geometric form and find the interval of convergence.

19. $f(x) = \frac{6}{1+x^2} = \frac{6}{1-(-x^2)}$

$6\sum_{n=0}^{\infty} (-x^2)^n = \boxed{\sum_{n=0}^{\infty} 6(-1)^n (x^{2n})}$

if $|-x^2| < 1$

$|x|^2 < 1$

$\boxed{(-1, 1)}$

20. $g(x) = \frac{x^3}{5+x} = \frac{x^3}{5} \cdot \frac{1}{1+\frac{x}{5}}$

$= \frac{x^3}{5} \cdot \frac{1}{1-\left(-\frac{x}{5}\right)} \rightarrow \frac{x^3}{5} \sum \left(-\frac{x}{5}\right)^n =$

$\sum \frac{1}{5}\left(\frac{1}{5}\right)^n (-1)^n (x)^{n+3} = \boxed{\sum \left(\frac{1}{5}\right)^{n+1} (-1)^n x^{n+3}}$

$\left|-\frac{x}{5}\right| < 1$

$|x| < 5$ $\boxed{(-5, 5)}$

$x=0$ $\quad (-1)^n \cdot 5^n$

$\sum \frac{(-1)^{n+1}(-5)^n}{n \cdot 5^n}$

$\sum \frac{(-1)^{2n+1}(5)^n}{n 5^n}$

$\sum \frac{-1}{n}$ div.

$x=10$

$\sum \frac{(-1)^{n+1} \cancel{5^n}}{n \cancel{5^n}}$

$\sum \frac{(-1)^{n+1}}{n}$ conv

$(0,10]$

$|x-5|<5$

$(0,10)$

$\frac{(2(n+1))!}{(2n+2)!}$

Find the interval of convergence

21. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n \cdot 5^n}$

$\lim_{n\to\infty} \left| \frac{(-1)^{n+2}(x-5)^{n+1}}{n+1 \cdot 5^{n+1}} \cdot \frac{n\, 5^n}{(-1)^{n+1}(x-5)^n} \right|$

$\lim \left| \frac{(-1)(x-5)\, \cancel{n}}{5(\cancel{n+1})} \right| = \left| \frac{x-5}{5} \right| < 1$

22. $\sum_{n=1}^{\infty} (2n)! \left(\frac{x}{2}\right)^n$

$\lim_{n\to\infty} \left| \frac{(2n+2)!\, x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(2n)!\, x^n} \right|$

$\lim_{n\to\infty} \left| \frac{(2n+2)(2n+1) \cdot x}{2} \right| = \infty > 1$

$\boxed{\text{converges only at } x=0}$

24. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

$\lim_{n\to\infty} \left| \frac{(-1)^{n+2}(x-1)^{n+2}}{n+2} \cdot \frac{n+1}{(-1)^{n+1}(x-1)^{n+1}} \right|$

$\lim_{n\to\infty} \left| \frac{(-1)(x-1)(n+1)}{(n+2)} \right| = |x-1| < 1$

$(0,2)$

$x=0$

$\sum \frac{(-1)^{n+1}(-1)^{n+1}}{n+1}$

$\sum \frac{1}{n+1}$

diverges LCT to $\sum \frac{1}{n}$

$x=2$

$\sum \frac{(-1)^{n+1}}{n+1}$ AST converges

$\lim \frac{1}{n+1} = 0$ ✓

$\frac{1}{n+2} < \frac{1}{n+1}$ dec ✓

$(0,2]$

Find the terms through degree four and the general term of the Maclaurin series for f(x).

25. $f(x)=\cos(4x^2)$

$\sum_{n=0}^{\infty} \frac{(-1)^n (4x^2)^{2n}}{(2n)!}$

$\sum \frac{(-1)^n 16^n x^{4n}}{(2n)!} = 1 - \frac{(4x^2)^2}{2!} + \frac{(4x^2)^4}{4!}$

26. $f(x)=x^2 e^{x^3}$

$x^2 \sum \frac{(x^3)^n}{n!} = \sum \frac{x^{3n+2}}{n!} =$

$x^2 \left(1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \cdots \right)$

$x^2 + x^5 + \frac{x^8}{2!} + \frac{x^{18}}{3!} + \cdots$

27. Express $\int_0^1 \tan^{-1}(x^2)\,dx$ as an infinite series and find its value to within an error of at most ~~0.1.~~ .01

$\int_0^1 \left(x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \frac{x^{18}}{9} - \cdots \right)$

$\left. \frac{x^3}{3} - \frac{x^7}{21} + \frac{x^{11}}{55} - \frac{x^{15}}{105} + \frac{x^{19}}{171} - \cdots \right]_0^1$

$\boxed{\frac{1}{3} - \frac{1}{21} + \frac{1}{55}} - \frac{1}{105} + \frac{1}{171}$

$\approx .303896$

.018

.009